Towards testing the Maldacena Conjecture with SDLCQ[1]

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Abstract

We consider the Maldacena conjecture applied to the near horizon geometry of a D1-brane in the supergravity approximation and present numerical results of a test of the conjecture against the boundary field theory calculation using supersymmetric discrete light-cone quantization (SDLCQ). We present numerical results with approximately 1000 times as many states as we previously considered. These results support the Maldacena conjecture and are within 10-15% of the predicted numerical results in some regions. Our results are still not sufficient to demonstrate convergence, and, therefore, cannot be considered to a numerical proof of the conjecture. We present a method for using a "flavor" symmetry to greatly reduce the size of the basis and discuss a numerical method that we use which is particularly well suited for this type of matrix element calculation.

1 Introduction

Recently, the conjecture has been put forth that certain field theories admit concrete realizations as string theories on particular backgrounds [1]. Attempts to rigorously test this so-called Maldacena conjecture have met with limited success, because our understanding of both sides of the correspondence is insufficient. The main obstacle is that at the point of correspondence, we require two conditions which are mutually exclusive. Namely, we want a situation where the curvature of the considered space-time is small, in order to be able to use the supergravity approximation to string theory. We also want the corresponding field theory to be in a small coupling regime. So far it has been impossible to find such a scenario. The solution to this paradox is to perform a non-perturbative calculation on the field theory side with a method that works optimal at the chosen point of correspondence.

Supersymmetric Discretized Light-Cone Quantization (SDLCQ) is a non-perturbative method for solving complicated bound-state problems that has been

[1] Based on work with S. Pinsky, O. Lunin, and J.R. Hiller.

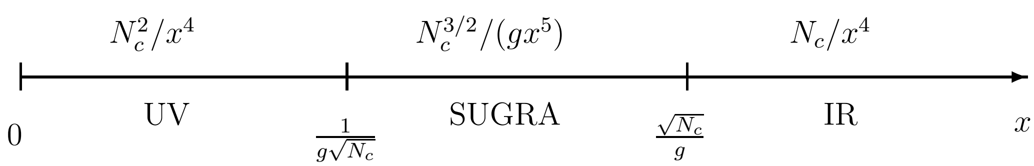
shown to have excellent convergence properties, in particular in low dimensions. The Yang-Mills theory with 16 supercharges in two dimensions [5] seems therefore a optimal candidate to study the field theory/string theory correspondence. Its corresponding string theory is a system of D1 branes in Type IIB string theory decoupling from gravity [4]. An observable that can be computed relatively easy on both sides of the correspondence is the correlation function of a gauge invariant operator, namely the stress-energy tensor $T^{\mu\nu}$. We will construct this observable in the supergravity approximation to string theory and perform a non-perturbative SDLCQ calculation of this correlator on the field theory side.

2 The Correlator from SUGRA

We can compute the two-point correlation function of the stress-energy tensor from string theory using the supergravity (SUGRA), *i.e.* small curvature, approximation [6–8]. Because of limited space, we cannot give any details of the calculation here. Essentially, one takes the near horizon geometry of a D1 brane in the string frame and asks for the action of fluctuations around this background. The diagonal fluctuation can be inferred from work on black hole absorption cross-sections. Solving the equations of motion for the lightest, *i.e.* dominant field, one can compute the flux factor. Its leading non-analytic term yields the correlator

$$\langle O(x)O(0)\rangle = \frac{N_c^{3/2}}{gx^5}. \tag{1}$$

As a consistency check we remark that the corresponding two-dimensional $N = (8,8)$ supersymmetric Yang-Mills theory has conformal fixed points in the ultraviolet and infrared with central charges N_c^2 and N_c, respectively. We expect to deviate from the trivial $(1/x^4)$ scaling behavior at $x_1 = \frac{1}{g\sqrt{N_c}}$ and $x_2 = \frac{\sqrt{N_c}}{g}$. This yields the following phase diagram:



3 The correlator from SDLCQ

Discretized Light-Cone Quantization (DLCQ) preserves supersymmetry at every stage of the calculation if the supercharge rather than the Hamiltonian is diagonalized [3]. The framework of supersymmetric DLCQ (SDLCQ) allows to use the advantages of light-cone quantization (*e.g.* a simpler vacuum) together with the excellent renormalization properties guaranteed by supersymmetry. Using SDLCQ, we can reproduce the SUGRA scaling relation, Eq. (1), fix the numerical coefficient, and calculate the cross-over behavior at $1/g\sqrt{N} < r < \sqrt{N}/g$. To exclude subtleties, *nota bene* issues of zero modes, we checked our results against the free fermion and the 't Hooft model and found consistent results.

We want to compute the correlator of the gauge invariant operator $T^{++}(-K)$

$$F(x^-, x^+) = \langle T(x^-, x^+) T(0,0) \rangle \quad ; \quad x^\pm \equiv \frac{1}{\sqrt{2}}(x^0 \pm x^1). \tag{2}$$

In DLCQ one fixes the total longitudinal momentum $P^+ = \frac{K\pi}{L}$, so we Fourier transform and spectrally decompose this quantity

$$\begin{aligned}
\tilde{F}(P^+, x^+) &= \frac{1}{2L} \langle T^{++}(P^+, x^+) T^{++}(-P^+, 0) \rangle \\
&= \sum_n \frac{1}{2L} \langle 0 | T^{++}(P^+) | n \rangle e^{-iP_+^n x^+} \langle n | T^{++}(-P^+) | 0 \rangle \\
&\equiv \left| \frac{L}{\pi} \langle n | T^{++}(-K) | 0 \rangle \right|^2 \frac{1}{2L} \left(\frac{\pi}{L} \right)^2 e^{-i \frac{M_n^2}{2P^+} x^+}
\end{aligned}$$

We can simplify the mixed representation by inverse Fourier transforming with respect to P^+

$$F(x^-, x^+) = \sum_n \left| \frac{L}{\pi} \langle n | T^{++}(-K) | 0 \rangle \right|^2 \left(\frac{x^+}{x^-} \right)^2 \frac{M_n^4}{8\pi^2 K^3} K_4 \left(M_n \sqrt{2x^+ x^-} \right) \tag{3}$$

and continue to Euclidean space by taking $r^2 = 2x^+ x^-$ to be real. This yields

$$C(r) = \left(\frac{x^-}{x^+} \right)^2 F(x^-, x^+) = \sum_n \left| \frac{L}{\pi} \langle n | T^{++}(-K) | 0 \rangle \right|^2 \frac{M_n^4}{8\pi^2 K^3} K_4(M_n r). \tag{4}$$

Note that this quantity depends on the harmonic resolution K, but involves no other unphysical quantities. In particular, the expression is independent of the box length L. We see that this result has the correct small r behavior

$$C(r) \quad \longrightarrow \quad \frac{(2n_b + n_f)}{4\pi^2} \left(1 - \frac{1}{K} \right) \frac{N_c^2}{r^4}, \tag{5}$$

which we expect for the theory of $n_b(n_f)$ free bosons (fermions) at large K.

In principle, we can now calculate the correlator numerically by evaluating Eq. (4). However, it turns out that even for very modest harmonic resolutions, we face a tremendous numerical task. At $K = 2, 3, 4$, the dimension of the associated Fock space is $256, 1632$, and 29056, respectively. Compared to previous work [2], we made the following improvements. Firstly, we rewrote the original Mathematica code into C++. Furthermore, we now exploit the discrete flavor symmetry of the problem to reduce the size of the Fock space by orders of magnitude. Finally, the numerical efficiency has been greatly improved by using Lanczos diagonalization techniques.

Let us first look at the discrete flavor symmetry. The theory has flavor symmetry, but we chose to diagonalize only one of the supercharges, Q_1^-. This complicates the symmetry structure of the problem significantly. However, there still exist symmetries S with $[P^-, S] = [T^{++}, S] = 0$, and $S|0\rangle = s_0|0\rangle$. The implementation of these symmetries will block-diagonalize P^- and reduce the numerical effort immensely. The form of supercharge is

$$
\begin{aligned}
Q_\alpha^- = & \int_0^\infty [...] b_\alpha^\dagger(k_3) a_I(k_1) a_I(k_2) + ... \\
& + (\beta_I \beta_J^T - \beta_J \beta_I^T)_{\alpha\beta} [...] b_\beta^\dagger(k_3) a_I(k_1) a_J(k_2) + ...,
\end{aligned}
$$

where β_I are 8×8 real matrices satisfying $\{\beta_I, \beta_J^T\} = 2\delta_{IJ}$. We thus have two flavor structures: the first first part of the supercharge proportional to $b_\alpha^\dagger a_I a_I$ is obviously invariant under S as long as $b_1 \to b_1$. The second part is more complicated, but it is possible to construct all transformations S which leave Q_1^- invariant. They form a subgroup of the permutation group $S_8 \times S_8$. We find seven Z_2 symmetries; they form a group of 168 elements. This means that we are able to reduce the size of the problem by a factor of (up to) 168! As an example, we list the first of the Z_2 symmetries

$$
\begin{aligned}
S_1 : & \; a_1 \to a_7, \quad a_2 \to a_3, \quad a_3 \to a_2, \quad a_4 \to a_6, \quad a_5 \to a_8, \quad a_6 \to a_4, \\
& \; a_7 \to a_1, \quad a_8 \to a_5, \quad b_2 \to b_2, \quad b_3 \to -b_3, \quad b_4 \to -b_4, \quad b_5 \to -b_6, \\
& \; b_6 \to -b_5, \quad b_7 \to b_8, \quad b_8 \to b_7
\end{aligned}
$$

To further reduce the numerical effort, we substitute the explicit diagonalization with an efficient approximation. The idea is to use a symmetry preserving (Lanczos) algorithm. If we start with a normalized vector $|u_1\rangle$ proportional to the fundamental state $T^{++}(-K)|0\rangle$, the Lanczos recursion will produce a tridiagonal representation of the Hamiltonian $H_{LC} = 2P^+P^-$. Due to orthogonality of $\{|u_i\rangle\}$, only the (1,1) element of the tridiagonal matrix, $\hat{H}_{1,1}$, will contribute to the correlator. We exponentiate by diagonalizing $\hat{H}_{LC}\vec{v}_i = \lambda_i \vec{v}_i$ with eigenvalues λ_i and obtain

$$
F(P^+, x^+) = \frac{1}{2L} \left(\frac{\pi}{L}\right)^2 \frac{1}{|N_0|^2} \sum_{j=1}^{N_L} |(v_j)_1|^2 e^{-i\frac{\lambda_j L}{2K\pi} x^+},
$$

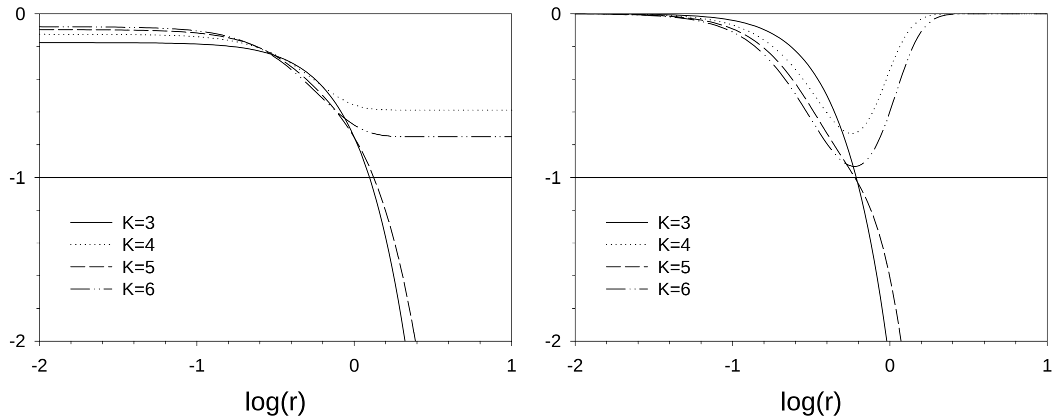


Fig. 1. Left: (a) Log-Log plot of $\langle T^{++}(x)T^{++}(0)\rangle \left(\frac{x^-}{x^+}\right)^2 \frac{4\pi^2 r^4}{N_c^2(2n_b+n_f)}$ v.s. r in units where $g_{YM}^2 N_c/\pi = 1$ for $K = 3, 4, 5$ and 6. Right: (b) the log-log derivative with respect to r of the correlation function in (a).

and finally we Fourier transform to obtain

$$F(x^-, x^+) = \frac{1}{8\pi^2 K^3}\left(\frac{x^+}{x^-}\right)^2 \frac{1}{|N_0|^2}\sum_{j=1}^{N_L} |(v_j)_1|^2 \lambda_j^2 K_4(\sqrt{2x^+x^-\lambda_i}),$$

which is equivalent to Eq. (4). This algorithm is correct only if the number of Lanczos iterations N_L runs up to the rank of of original matrix. But *in praxi* already a basis of about 20 vectors covers all leading contribution to correlator [9].

4 Results

To evaluate expression for the correlator $C(r)$, we have to calculate the mass spectrum and insert it into Eq. (4). We consider $N = (8, 8)$ supersymmetric Yang-Mills theory [5], conjectured to be equivalent to the system of D1 branes, as described above. Here, the contribution of massless states become a real problem. These states exist in the SDLCQ calculation, but are unphysical. It can be shown that theses states are not normalizable and that the number of partons in these states is even/odd for K even/odd. Because the correlator is only sensitive to two particle contributions, the curves $C(r)$ are different for even/odd K. Unfortunately, the unphysical states yield also the typical $1/r^4$ behavior, but have a wrong N_c dependence. The regular $1/r^4$ contribution is down by $1/N_c$, so we cannot see this contribution at large r, because we are working in the large N_c limit. We leave however the unphysical in the calculation, because they help us to determine when our approximation breaks down. The calculations are consistent in the sense that this breakdown occurs at larger and larger r as K grows. We expect to approach the line

$dC(r)/dr = -1$ line signaling the cross-over from the trivial $1/r^4$ behavior to the characteristic $1/r^5$ behavior of the SUGRA correlator, Eq. (1). We see from Fig. 3, that we actually get very close to a slope of -1, before the approximation breaks down. A safe signature of equivalence of the field and string theories would be if the derivative curve flattens at -1 before approximation breaks down.

5 Conclusions

In this note we reported on progress in an attempt to rigorously test the conjectured equivalence of $N = (8, 8)$ supersymmetric Yang-Mills theory and a system of $D1$ branes in string theory. Within a well-defined non-perturbative calculation, we obtained results that are within 10-15% of results expected from the Maldacena conjecture. The results are still not conclusive, but they definitely point in right direction. Compared to previous work [2], we included a factor 100-1000 more states in our calculation and thus greatly improved the testing conditions. We remark that improvements of the code and the numerical method are possible and under way. During the calculation we noticed that contributions to the correlator come from only a small number of terms. An analytic understanding of this phenomenon would greatly accelerate calculations. We remark that in principal we could study the proper $1/r$ behavior at large r by computing $1/N_c$ corrections, but this interesting calculation would mean a huge numerical effort.

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